Indag. Mathem., N.S., 4 (2), 203-210 June 21, 1993

Expansiveness, entropy and polynomial growth for groups acting on subshifts by automorphisms

by Mark A. Shereshevsky*

Mathematics Institute, University of Warwick, Coventry, CV4 7AL, England

Communicated by Prof. M.S. Keane at the meeting of September 21, 1992

ABSTRACT

Let X be a closed translationally invariant subset of the d-dimensional full shift $P^{\mathbb{Z}^d}$, where P is a finite set, and suppose that the $\mathbb{Z}^d$-action on X by translations has positive topological entropy. Let G be a finitely generated group of polynomial growth. We prove that if growth$(G)<d$, then any G-action on X by homeomorphisms commuting with translations is not expansive. On the other hand, if growth$(G)=d$, then any expansive G-action on X by homeomorphisms commuting with translations has positive topological entropy. Analogous results hold for semigroups.

1. INTRODUCTION

The study of automorphisms and endomorphisms (i.e. continuous shift commuting maps, invertible or non-invertible) of the full shift and its subshifts was begun by Hedlund and coworkers [7] and Coven and Paul [2]. In this note we consider an actions of a finitely generated group G of polynomial growth by automorphisms of a subshift of the d-dimensional full shift. If the subshift has positive topological entropy, we find that the degree of polynomial growth of the G may strongly affect certain dynamical properties of the action. More precisely, if growth$(G)<d$, then the action cannot be expansive. On the other hand, if the growth(G) is exactly d, then an expansive action must have positive topological entropy. The results still hold for finitely generated semigroups of polynomial growth acting by endomorphisms on a subshift with positive entropy. In particular, this implies that, firstly, there are no expansive endo- (or

* Present address: MSRI, 1000 Centennial Drive, Berkeley, CA 94720, USA

auto-)morphisms of such subshifts with $d>1$ and, secondly, that if $d=1$, then any expansive endo- (or auto-)morphism has positive entropy.

The obstruction to expansiveness we have obtained here supplements, to some extent, the following fact established by A. Fathi.

Proposition ([4, Corollary 5.6]). If a compact topological space admits an expansive homeomorphism with zero entropy, then its topological dimension is zero.

2. PRELIMINARIES

Subshifts. Let P be a finite set with cardinality $\mathrm{Card}(P)>1$. For any $d\geq 1$ we put

$$\Omega_d(P)=\{x=(x_n)_{n\in\mathbb{Z}^d}: x_n\in P \text{ for all } n\in\mathbb{Z}^d\}.$$

In other words, $\Omega_d(P)$ is just the set $P^{\mathbb{Z}^d}$ of all maps $x:\mathbb{Z}^d\to P$. Provide $\Omega_d(P)$ with the product topology. Then $\Omega_d(P)$ is a compact Hausdorff zero-dimensional space. Define, for every $m\in\mathbb{Z}^d$, a homeomorphism $\tau^m:\Omega_d(P)\to\Omega_d(P)$ by setting $(\tau^m x)_n=x_{n+m}$ for all $x\in\Omega_d(P)$ and $n\in\mathbb{Z}^d$. The correspondence $m\mapsto\tau^m$ defines a continuous $\mathbb{Z}^d$-action τ on $\Omega_d(P)$. The pair (X,τ_X), where X is a closed τ-invariant subset of $\Omega_d(P)$ and $\tau_X=\tau\,|\,X$ is the restriction of the $\mathbb{Z}^d$-action τ to X, is called a *subshift*. A continuous map $T: X\to X$ commuting with τ (i.e. $T\circ\tau^m=\tau^m\circ T$ for all $m\in\mathbb{Z}^d$) is called an *automorphism of the subshift* if T is invertible. If T is non-invertible it is called an *endomorphism of the subshift.*

Expansiveness. Let X be a compact topological space and let G be a group acting on X by homeomorphisms. We denote the action by T and the homeomorphism corresponding to $g\in G$ by T^g. The action T is said to be *expansive*, if there is a closed neighbourhood $V\subset X\times X$ of the diagonal $\Delta_X=\{(x,y)\in X\times X: x=y\}$ such that for $\hat{T}=T\times T: X\times X\to X\times X$ we have $\bigcap_{g\in G}\hat{T}^g V=\Delta_X$. If the space X is equipped with a metric ϱ, then expansiveness means that there is an *expansive constant* $c>0$ such that

$$x,y\in X,\quad x\neq y \text{ implies } \varrho(T^g x, T^g y)>c \text{ for some } g\in G.$$

In a similar way one defines (positive) expansiveness for semigroups acting on a compact space by (non-invertible) continuous maps.

Entropy (see [9], [5] for details). It is known that a countable group G is *amenable* if and only if it contains a sequence of finite subsets $A_1\subset A_2\subset\cdots\subset A_n\subset\cdots\subset G$ with the properties:

(i) $\bigcup_{n\geq 1} A_n=G$;

(ii) $\lim_{n\to\infty}|gA_n\triangle A_n|/|A_n|=0$ for every $g\in G$.

Such a sequence is called *Følner sequence. Let* T be a continuous action of an amenable group G on a compact topological space X. For a finite open cover ξ of X and a finite set $A\subset G$ let $\xi^A=\bigvee_{g\in A}(T^g)^{-1}\xi$. For a finite open cover η of X let $\mathcal{N}(\eta)$ stand for the minimal cardinality of subcovers of η. Choose a Følner sequence $\{A_n\}_{n\geq 1}$ in G and a finite open cover ξ of X and define.

$$h(T,\xi)=\limsup_{n\to\infty}|A_n|^{-1}\log\mathcal{N}(\xi^{A_n}).$$

This value does not depend on the particular Følner sequence $\{A_n\}_{n\ge 1}$. Then the topological entropy of the action T is defined as

$$h(T)=\sup\{h(T,\xi): \xi \text{ is a finite open cover of } X\}.$$

The *topological entropy* for continuous actions of amenable semigroups is defined analogously.

Groups of polynomial growth (see [1], [6] for details). Consider a group G generated by a finite subset $F\subset G$. Define the norm $\|g\|_F$ of an element $g\in G$ with respect to F to be the least integer $l\ge 0$ such that g can be expressed as a product $f_1 f_2 \cdots f_l$ with each $f_i\in F\cup F^{-1}$. We denote by $B_F(m)$, $m\ge 1$ the 'closed' ball of radius m, i.e. $B_F(m)=\{g\in G: \|g\|_F\le m\}$ and set $\beta_F(m)=\operatorname{Card}(B_F(m))$. Following [1], we say that G has *polynomial growth of degree k* if there exist constants $A, C>0$ such that $Am^k\le\beta_F(m)\le Cm^k$ for all $m\ge 1$. It is easily seen that this notion does not depend on the choice of F. In this case we write $\operatorname{growth}(G)=k$. One can show that if G has polynomial growth, then the balls $B_F(m)$, $m=1,2,\dots$ form a Følner sequence in G, and, therefore, G is amenable.

For instance, it was proved in [1] (see also [6]) that if G is a finitely generated nilpotent group with lower central series $G=G_1\supset G_2\supset\cdots\supset G_{p+1}=\{e\}$, then $\operatorname{growth}(G)=\sum_{q\ge 1} qr_q$, where $r_q=\operatorname{rank}(G_q/G_{q+1})$. Clearly, $\operatorname{growth}(\mathbb{Z}^d)=d$.

In the same manner the degree of polynomial growth can be defined for finitely generated semigroups.

3. RESULTS

Let G be a finitely generated (semi-)group with $\operatorname{growth}(G)=k$ and $X\subset\Omega_d(P)$ be a subshift with $h(\tau_X)>0$.

Theorem 1. *If $k=d$, then for any expansive action T of the (semi)group G by automorphisms (endomorphisms) on the subshift X we have $h(T)>0$.*

Theorem 2. *If $k<d$, then any action T of the (semi)group G by automorphisms (endomorphisms) on the subshift X is not expansive.*

In the case $G=\mathbb{Z}^k$ (or $\mathbb{Z}^k_+$) the theorems above describe dynamical properties of the joint action of k commuting automorphisms (endomorphisms) of the subshift $X\subset\Omega_d(P)$ according as $k=d$ or $k<d$. In particular for $G=\mathbb{Z}$ we have

Corollary 1. *Let $d=1$ and $X\subset\Omega_1(P)$ be a subshift with $h(\tau_X)>0$. Let T be an expansive endomorphism or automorphism of X. Then $h(T)>0$.*

Corollary 2. *Let $d>1$ and $X\subset\Omega_d(P)$ be a subshift with $h(\tau_X)>0$. Then there are neither expansive endomorphisms nor expansive automorphisms of X.*

We remark that if we drop the condition that the entropy of the subshift X be positive, then the last statement is no longer true. To give a counter example consider the 2-dimensional subshift of finite type

$$X=\{x\in\Omega_d(\mathbb{Z}/2\mathbb{Z}): x_{(i,j)}+x_{(i+1,j)}+x_{(i,j+1)}=0 \ (\mathrm{mod}\ 2), \forall (i,j)\in\mathbb{Z}^2\}.$$

This subshift was first studied by Ledrappier [8] (see also [10]) and was shown to have zero entropy ($h(\tau_X)=0$). On the other hand one readily verifies that the automorphism $T=\tau^{(1,1)}\mid X$ and the endomorphism S, defined by $(Sx)_{(i,j)}=x_{(i-1,j-1)}+x_{(i,j)}+x_{(i+1,j+1)}$ are expansive.

Also, it should be mentioned that our results can be reformulated as follows. Let Y be an arbitrary compact zero-dimensional topological space and $\boldsymbol{m}\mapsto\sigma^{\boldsymbol{m}}$ be a continuous expansive $\mathbb{Z}^k$-action on Y. Then one can construct a finite topological generator (cf. [3], [11]) $\mathscr{U}=\{U_i\}$ with $U_i\cap U_j=\emptyset$ for $i\neq j$, by means of which we define (in the standard way) the topological conjugacy between (Y,τ) and a d-dimensional subshift (X,τ_X). Suppose $h(\sigma)>0$ and let G again be a finitely generated group with growth$(G)=k$. From Theorems 1 and 2, respectively, we have.

Proposition 1. *Suppose $k=d$. Let T be an expansive continuous action of the (semi-)group G on Y commuting with σ. Then $h(T)>0$.*

Proposition 2. *Suppose $k<d$. Then any continuous action of the (semi-)group G on Y commuting with σ is not expansive.*

4. PROOFS OF THEOREMS 1 AND 2

The proofs will be carried out for the case where G is a group, the 'non-invertible' case where G is a semigroup is treated similarly.

First we prove the following technical result.

Lemma 1 (cf. [3]; p. 109). *Let T be an expansive continuous action of a group G on a compact metric space (X,ϱ) and let $c>0$ be an expansive constant for T. Let $A_1\subset A_2\subset\cdots\subset A_n\subset\cdots\subset G$ be a sequence of subsets of G such that $\bigcup_{n\geq 1}A_n=G$. Then for any $\delta>0$ there exists an integer $M=M(\delta)\geq 1$ such that $\varrho(x,y)\geq\delta$ implies existence of some $g\in A_M$ satisfying $\varrho(T^gx,T^gy)>c$.*

PROOF. Suppose, on the contrary that for some $\delta>0$ there exist sequences $x_n, y_n\in X$, $n=1,2,\ldots$ satisfying $\varrho(x_n,y_n)\geq\delta$ but with $\varrho(T^gx_n,T^gy_n)\leq c$ for all $g\in A_n$. By compactness the sequence $(x_n,y_n)\in X^2$ contains a convergent subsequence. To avoid double indices we assume that the sequence (x_n,y_n) itself converges to a pair $(x,y)\in X^2$. Take any $g\in G$. Then $g\in A_n$, and hence $\varrho(T^gx_n,T^gy_n)\leq c$, for all sufficiently large n. Since the action is continuous, we have $\varrho(T^gx,T^gy)\leq c$. The last inequality holds for every $g\in G$ which, in view of expansiveness implies $x=y$. But this is impossible, because we must have $\varrho(x,y)\geq\delta$. The contradiction proves the lemma. □

Now we provide the topological space $\Omega_d(P)$ with a metric. For $\boldsymbol{n}=(n_1,\ldots,n_d)$ we put $|\boldsymbol{n}|=\max_{1\leq i\leq d}|n_i|$. Let α be a fixed real number with

$0<\alpha<1$ and put $L(x,y)=\min\{|n|: x_n \neq y_n\}$ for $x\neq y$. We define the metric ϱ_α on $\Omega_d(P)$ by putting

$$P_\alpha(x,y) = \begin{cases} \alpha^{L(x,y)}, & \text{if } x\neq y, \\ 0, & \text{if } x=y. \end{cases}$$

Every subshift $X\subset\Omega_d(P)$ is given the induced metric.

Then we introduce a family of partitions of X which will be of substantial use in the proofs of the main results.

Given integers $a_i\leq b_i$, $1\leq i\leq d$ define the rectangle

$$R(\boldsymbol{a},\boldsymbol{b})=\{\boldsymbol{n}\in\mathbb{Z}^d: a_i\leq n_i\leq b_i \text{ for all } 1\leq i\leq d\},$$

$\boldsymbol{a}=(a_1,\dots,a_d)$, $\boldsymbol{b}=(b_1,\dots,b_d)$. We introduce the partition $\xi(\boldsymbol{a},\boldsymbol{b})$ of the space $\Omega_d(P)$ into compact open sets of the form

$$C(\boldsymbol{a},\boldsymbol{b};\hat{u})=\{x\in\Omega_d(P): x_n=u_n \text{ for all } \boldsymbol{n}\in R(\boldsymbol{a},\boldsymbol{b})\},$$

where $\hat{u}=(u_n)_{n\in R(\boldsymbol{a},\boldsymbol{b})}\in P^{R(\boldsymbol{a},\boldsymbol{b})}$. The partition $\xi(\boldsymbol{a},\boldsymbol{b})$ induces a partition of the subshift $X\subset\Omega_d(P)$ consisting of the non-empty intersections of X with elements of $\xi(\boldsymbol{a},\boldsymbol{b})$. We denote this partition of X by $\xi_X(\boldsymbol{a},\boldsymbol{b})$ and its elements by $C_X(\boldsymbol{a},\boldsymbol{b};\hat{u})$. Note that the partition may also be considered as an open cover.

Let $\{R_k=R(\boldsymbol{a}^{(k)},\boldsymbol{b}^{(k)})\}_{k=1,2,\dots}$ be an increasing sequence of finite rectangles with $R_k\nearrow\mathbb{Z}^d$. Then, the topological entropy of the $\mathbb{Z}^d$-action τ_X on the subshift X is given by the formula (see [10])

$$h(\tau_X)=\lim_{k\to\infty}(\mathrm{Card}(R_k))^{-1}\log\mathrm{Card}(\xi_X(\boldsymbol{a}^{(k)},\boldsymbol{b}^{(k)})), \tag{1}$$

where the limit always exists and does not depend on the sequence $\{R_k\}$.

In what follows we write $\zeta\leq\eta$ for the partitions ζ and η, if η is finer than ζ and $\bigvee_{j\in J}\eta_j$ for the refinement of the partitions $\{\eta_j\}_{j\in J}$.

Let G be a finitely generated group with a fixed finite system of generators F. The proofs of Theorems 1 and 2 are based on the following.

Lemma 2. *Let $X\subset\Omega_d(P)$ be a subshift and T be an expansive action of the group G by automorphisms on X with the expansive constant α^{r+1} with some $r\in\mathbb{Z}_+$. Then there exists an integer $M\geq 1$ such that*

$$\bigvee_{g\in B_F(M)}(T^g)^{-1}\xi_X(-\boldsymbol{m},\boldsymbol{m})\geq\xi_X(-\boldsymbol{m}-\boldsymbol{1},\boldsymbol{m}+\boldsymbol{1}) \tag{2}$$

for all $m\geq r$ (where $\boldsymbol{m}=(m,\dots,m)$, $\boldsymbol{1}=(1,\dots,1)$).

PROOF. Let us apply Lemma 1 to the action T on the subshift X with the metric ϱ_α taking $\delta=c=\alpha^{r+1}$ and $A_n=B_F(n)$, $n=1,2,\dots$. Put $M=M(\delta)$. Choose arbitrarily two points $x,y\in X$ lying in different elements of $\xi_X(-\boldsymbol{r}-\boldsymbol{1},\boldsymbol{r}+\boldsymbol{1})$. Then $\varrho(x,y)\geq\alpha^{r+1}$ and, by Lemma 1, we have $\varrho(T^gx,T^gy)>\alpha^{r+1}$ for some $g\in B_F(M)$. This means that T^gx and T^gy lie in different elements of the partition $\xi_X(-\mathbf{r},\mathbf{r})$. Hence, x and y are in different elements of

$$\bigvee_{g \in B_F(M)} (T^g)^{-1} \xi_X(-\mathbf{r},\mathbf{r}),$$

whenever they are in different elements of $\xi_X(-\mathbf{r}-\mathbf{1},\mathbf{r}+\mathbf{1})$. This implies

$$\bigvee_{g \in B_F(M)} (T^g)^{-1} \xi_X(-\mathbf{r},\mathbf{r}) \geq \xi_X(-\mathbf{r}-\mathbf{1},\mathbf{r}+\mathbf{1}).$$

Now we prove (2) by induction on $\mathbf{m}$. Suppose that (2) holds for some $\mathbf{m} \geq \mathbf{r}$. Then, since $\tau X = X$, and $T \circ \tau = \tau \circ T$, we have

$$\bigvee_{g \in B_F(M)} (T^g)^{-1} \xi_X(-\mathbf{m}+\mathbf{j},\mathbf{m}+\mathbf{j}) \geq \xi_X(-\mathbf{m}+\mathbf{j}-\mathbf{1},\mathbf{m}+\mathbf{j}+\mathbf{1})$$

for every $\mathbf{j} \in \mathbb{Z}^d$. From this it follows that

$$(3) \qquad \begin{cases} \displaystyle\bigvee_{g \in B_F(M)} (T^g)^{-1} \xi_X(-\mathbf{m}-\mathbf{1},\mathbf{m}+\mathbf{1}) \geq \bigvee_{g \in B_F(M)} (T^g)^{-1} \xi_X(-\mathbf{m}+\mathbf{j},\mathbf{m}+\mathbf{j}) \\ \qquad\qquad\qquad\qquad \geq \xi_X(-\mathbf{m}+\mathbf{j}-\mathbf{1},\mathbf{m}+\mathbf{j}+\mathbf{1}). \end{cases}$$

for every $\mathbf{j} \in \mathbb{Z}^d$ with $|\mathbf{j}| = 1$. Obviously,

$$R(-\mathbf{m}-\mathbf{2},\mathbf{m}+\mathbf{2}) = \bigcup_{|\mathbf{j}|=1} R(-\mathbf{m}+\mathbf{j}-\mathbf{1},\mathbf{m}+\mathbf{j}+\mathbf{1})$$

and, therefore, $\xi_X(-\mathbf{m}-\mathbf{2},\mathbf{m}+\mathbf{2}) = \bigvee_{|\mathbf{j}|=1} \xi_X(-\mathbf{m}+\mathbf{j}-\mathbf{1},\mathbf{m}+\mathbf{j}+\mathbf{1})$. From this and (3) we have

$$\bigvee_{g \in B_F(M)} (T^g)^{-1} \xi_X(-\mathbf{m}-\mathbf{1},\mathbf{m}+\mathbf{1}) \geq \xi_X(-\mathbf{m}-\mathbf{2},\mathbf{m}+\mathbf{2}).$$

So, we have derived (2) for $m+1$ which completes the proof. □

Lemma 2 can be strengthened as follows.

Lemma 3. *Under the conditions of Lemma* 2 *we have*

$$(4) \qquad \bigvee_{g \in B_F(sM)} (T^g)^{-1} \xi_X(-\mathbf{m},\mathbf{m}) \geq \xi_X(-\mathbf{m}-\mathbf{s},\mathbf{m}+\mathbf{s})$$

for all $\mathbf{m} \geq \mathbf{r}$ *and all* $s \geq 1$ (*where* $\mathbf{s} = (s, \ldots, s)$).

PROOF. First we observe that for any $a, b \in \mathbb{N}$ we have

$$B_F(a) B_F(b) = \{g_1 g_2 \in G \colon g_1 \in B_F(a), g_2 \in B_F(b)\} = B_F(a+b),$$

in particular, $B_F((s+1)M) = B_F(sM) B_F(M)$. Using Lemma 2 we write

$$\begin{aligned} &\bigvee_{g \in B_F((s+1)M)} (T^g)^{-1} \xi_X(-\mathbf{m},\mathbf{m}) \\ &\quad = \bigvee_{g \in B_F(sM)} (T^g)^{-1} \Big(\bigvee_{h \in B_F(M)} (T^h)^{-1} \xi_X(-\mathbf{m},\mathbf{m}) \Big) \\ &\quad \geq \bigvee_{g \in B_F(sM)} \xi_X(-\mathbf{m}-\mathbf{1},\mathbf{m}+\mathbf{1}), \end{aligned}$$

and (4) follows by induction on s. □

Now the proofs of the main theorems are almost immediate.

PROOF OF THEOREM 1. Let G be a group generated by a finite set $F \subset G$ with $\mathrm{growth}(G) = d$ and $X \subset \Omega_d(P)$ be a subshift with $h(\tau_X) > 0$. Let T be an expansive action of G by automorphisms on X. We have $Am^d \leq \beta_F(m) \leq Cm^d$, $m \geq 1$ with some constants $A, C > 0$, where $\beta_F(m) = \mathrm{Card}(B_F(m))$. Using Lemma 3 and (1) we find that for any $m \geq r$

$$\begin{aligned} & h(T, \xi_X(-m,m)) \\ & \geq \limsup_{s \to \infty} \beta_F(sN)^{-1} \log \mathcal{N}\Big(\bigvee_{g \in B_F(sN)} (T^g)^{-1} \xi_X(-m,m)\Big) \\ & \geq (2/M)^d \lim_{s \to \infty} ((sM)^d / \beta_F(sM)) \\ & \quad \times (2(m+s)+1)^{-d} \log \mathcal{N}(\xi_X(-m-s, m+s)) \geq (2/M)^d C^{-1} h(\tau_X). \end{aligned}$$

Hence, $h(T) \geq (2/M)^d C^{-1} h(\tau_X)$ and $h(\tau_X) > 0$ implies $h(T) > 0$. □

PROOF OF THEOREM 2. Let G be a group generated by a finite set $F \subset G$ with $\mathrm{growth}(G) = k < d$ and $X \subset \Omega_d(P)$ be a subshift with $h(\tau_X) > 0$. Suppose T is an expansive action of G by automorphisms on X. Using Lemma 3 again we obtain that for any $m \geq r$

$$\begin{aligned} & h(T, \xi_X(-m,m)) \\ & \geq \limsup_{s \to \infty} \beta_F(sN)^{-1} \log N\Big(\bigvee_{g \in B_F(sN)} (T^g)^{-1} \xi_X(-m,m)\Big) \\ & \geq (2/M)^d \lim_{s \to \infty} ((sM)^d / \beta_F(sM)) \\ & \quad \times (2(m+s)+1)^{-d} \log \mathcal{N}(\xi_X(-m-s, m+s)). \end{aligned}$$

Thus, we have $h(T, \xi_X(-m,m)) = \infty$, since $\mathrm{growth}(G) < d$ implies $(sM)^d / \beta_F(sM) \to \infty$ and $(2(m+s)+1)^{-d} \log \mathcal{N}(\xi_X(-m-s, m+s)) \to h(\tau_X) > 0$. But this is impossible in view of the obvious inequality $h(T, \eta) \leq \log \mathrm{Card}(\eta)$ which holds for any finite open cover η. This contradiction completes the proof. □

ACKNOWLEDGEMENT

I am thankful to Klaus Schmidt for stimulating discussions.

REFERENCES

1. Bass, H. – The degree of polynomial growth of finitely generated nilpotent groups, Proc. London Math. Soc. **25**, 603–614 (1972).
2. Coven, E.M. and M. Paul – Endomorphisms of irreducible subshifts of finite type, Math. Syst. Theory **8**, 167–175 (1974).
3. Denker, M., C. Grillenberger and K. Sigmund – Ergodic Theory on Compact Spaces, Lecture Notes in Math. No. 527, Springer-Verlag, Berlin, 1976.
4. Fathi, A. – Expansiveness, hyperbolicity and Hausdorff dimension, Commun. Math. Phys. **126**, 242–261 (1989).
5. Feldman, J. – Ergodic theory of continuous group actions, Astérisque **49**, 61–74 (1977).

6. Gromov, M. – Groups of polynomial growth and expanding maps, Publ. Math. IHES **53**, 53–73 (1981).
7. Hedlund, G.A. – Endomorphisms and automorphisms of the shift dynamical systems, Math. Syst. Theory **3**, 320–375 (1969).
8. Ledrappier, F. – Un champ markovien peut être d'entropie nulle et mélangeant, C.R. Acad. Sci. Paris, Ser. A, **287**, 561–562 (1987).
9. Moulin-Ollagnier, J. – Ergodic Theory and Statistical Mechanics, Lecture Notes in Math. No. 1115, Springer-Verlag, Berlin, 1985.
10. Schmidt, K. – Algebraic Ideas in Ergodic Theory, CBMS Regional Conf. Ser. in Math., No. 76, Amer. Math. Soc., Providence RI, 1990.
11. Walters, P. – An Introduction to Ergodic Theory, Springer-Verlag, Berlin, 1982.